Effective integration of Lie type algebras

Joint work with V. Dotsenko & S. Shadrin : 1502.03280
2212.11323

D. Robert-Nicoud : 2010.10485

R. Campos : to appear

I - Introduction

- Sophus Lie (1842–1899, Norway) : "Third theorem"

$$\text{Lie groups} \xleftrightarrow{\cong} \text{Lie algebras}$$

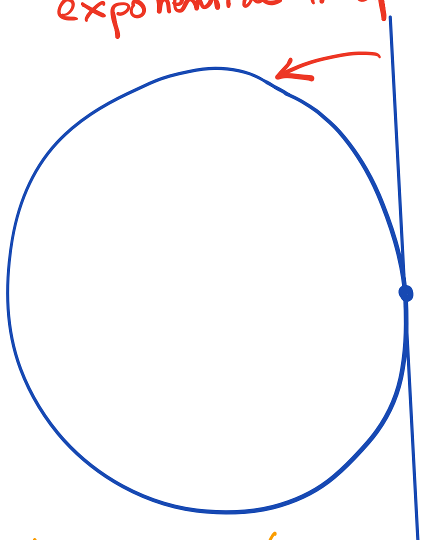
(real, simply connected) infinitesimal ⟶ (real, finite dimensional)

global ⟵ integration local

exponential map

$\mathfrak{g} = \mathbb{R}$
$= T_e G$

$G = S^1$



product: universal formula

Baker-Campbell-Hausdorff (BCH)

- Ludwig Maurer (1859–1927) & Élie Cartan (1869–1951)

→ Study the integrability of Lie algebras :

Maurer-Cartan equation : $\boxed{d\omega + \frac{1}{2}[\omega, \omega] = 0}$

• <u>Deformation theory</u> (1960's) underlying "space" A

type of structure $\mathcal{P}$

Heuristic: $\exists\ (\mathfrak{g}_{\mathcal{P},A}, d, [\,,\,])$ differential graded Lie algebra

$$\mathcal{P}\text{-structures on } A \overset{1-1}{\longleftrightarrow} MC(\mathfrak{g}_{\mathcal{P},A}) := \left\{ \alpha \in (\mathfrak{g}_{\mathcal{P},A})_{-1} \,;\, d\alpha + \frac{1}{2}[\alpha,\alpha] = 0 \right\}$$

<u>Maurer-Cartan elements</u>

equivalence relation $\overset{1-1}{\longleftrightarrow}$ $G :=$ group integrating the Lie algebra $(\mathfrak{g}_{\mathcal{P},A})_0$

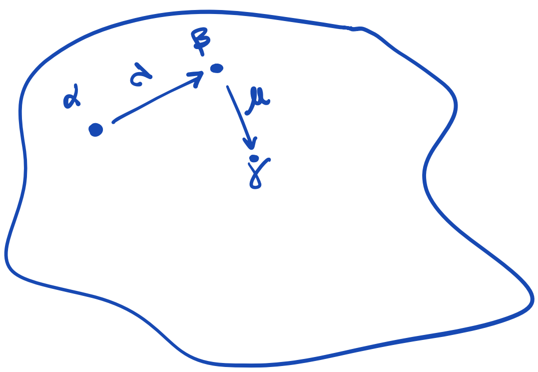
<u>gauge group.</u>

↳ <u>Deligne groupoid</u>:

• objects: $\alpha, \beta, \gamma, \ldots \in MC(\mathfrak{g}_{\mathcal{P}A})$

• Morphisms: $\lambda \in (\mathfrak{g}_{\mathcal{P},A})_0$

$$\alpha \xrightarrow{\lambda} \beta = \lambda \cdot \alpha$$



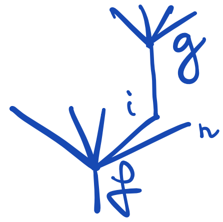
• <u>Example</u>: A vector space, $\mathcal{P} = \mathrm{ass}$: associative algebra structure

$$Y : A^{\otimes 2} \longrightarrow A \text{ s.t. } Y = Y$$

equivalence = isomorphisms $(\mathbb{k} = \mathbb{C})$

$$\longrightarrow \mathfrak{g}_{\mathrm{ass},A} = \prod_{n \geq 1} \mathrm{Hom}(A^{\otimes n}; A)$$ "Hochschild cochain cx"

degree $:= 1-n$

$$f \star g := \sum_{i=1}^{n} \pm 1$$  $$\xrightarrow[\text{symmetrization}]{\text{skew-}} [f,g] := \underbrace{f \star g - \pm g \star f}$$

pre-Lie product $\Longrightarrow$ Lie bracket (Jacobi identity)

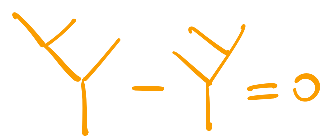
$\mathrm{Assoc}(f, \underline{g}, \underline{h}) := (f \star g) \star h - f \star (g \star h)$

$= \pm \mathrm{Assoc}(f, \underline{h}, \underline{g})$

right symmetric

Gerstenhaber bracket

$$\boxed{\begin{array}{l} |\alpha| = -1 \longleftrightarrow \alpha : A^{\otimes 2} \longrightarrow A \\ \underbrace{d\alpha}_{\overset{\shortparallel}{0}} + \frac{1}{2} \underbrace{[\alpha, \alpha]}_{\overset{\shortparallel}{2\alpha \star \alpha}} = 0 \Leftrightarrow \alpha \star \alpha = 0 : \end{array}}$$



2010

• Fundamental theorem of deformation theory [Pridham–Lurie]
[Le Grignou – Roca Lucio 2023]

char$(\mathbb{K}) = 0$ Deformation problems $\overset{1-1}{\longleftrightarrow}$ dg Lie algebras

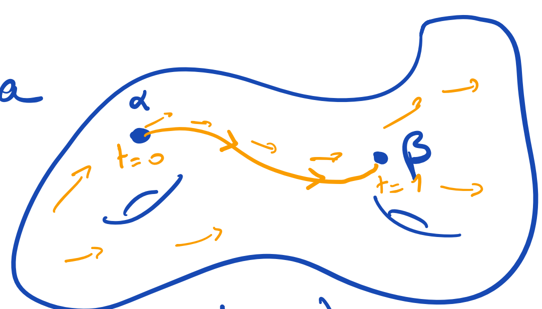
Formal moduli problems $\underset{\substack{\text{equivalence of} \\ \infty\text{-categories}}}{\simeq}$ dg Lie algebras.

effective formulas ...

II _ The classical case

$(\mathfrak{g}, d, [\,,\,])$: (nilpotent, complete) dg Lie algebra

$\lambda \in \mathfrak{g}_0 \longmapsto d\lambda + \underbrace{[\lambda, -]}_{= \mathrm{ad}_\lambda} \in \Gamma(T\,\mathrm{MC}(\mathfrak{g}))$: vector field



$\mathrm{MC}(\mathfrak{g})$

Definition: $\alpha \sim \beta$ gauge equivalent $\qquad \downarrow$ integration

$$\boxed{\exists \lambda \in \mathfrak{g}_0 \text{ s.t. } \dot{\gamma}(t) = d\lambda + [\lambda, \gamma(t)] \text{ satisfies } \gamma(0)=\alpha,\ \gamma(1)=\beta}$$

Question 1: $\gamma(1) = ?$

when $d=0$ $\quad \boxed{\gamma(1) = e^{\mathrm{ad}_\lambda}(\alpha) =: \lambda.\alpha}$

$\rightarrow$ no loss of generality: $\mathfrak{g} \hookrightarrow \mathfrak{g} \oplus \mathbb{K}\delta$: differential trick

$d=0 \qquad |\delta| = -1$

$[\delta, x] := dx$

Question 2: $\sim$ equivalence relation ?

yes if a group action: $\lambda.(\mu.\alpha) = e^{\mathrm{ad}_\lambda}(e^{\mathrm{ad}_\mu}(\alpha)) \overset{?}{=} e^{\mathrm{ad}_{\lambda * \mu}}(\alpha)$

$\rightarrow$ particular case: $[a,b] = \underbrace{a \star b - b \star a}_{\text{associative (unital)}}$

action: $\boxed{\lambda.\alpha = e^{\lambda} \star \alpha \star e^{-\lambda}}$

$$\lambda.(\mu.\alpha) = e^{\mathrm{ad}_\lambda}(e^{\mathrm{ad}_\mu}(\alpha)) = e^{\lambda} \star e^{\mu} \star \alpha \star e^{-\mu} \star e^{-\lambda} = e^{\mathrm{ad}_{\lambda * \mu}}(\alpha)$$

for group product: $\boxed{\lambda * \mu = \ln(e^{\lambda} \star e^{\mu})}$

$\rightarrow$ no loss of generality:

Theorem [Baker-Campbell-Hausdorff, 1905].

$$\ln\left(e^{\lambda}e^{\mu}\right) \in \widehat{\mathcal{L}ie}(\lambda,\mu) \subset \widehat{Ass}(\lambda,\mu)$$

$$\overset{!!}{\mathrm{BCH}(\lambda,\mu)} = \lambda+\mu+\frac{1}{2}[\lambda,\mu]+\frac{1}{12}[\lambda,[\lambda,\mu]]+\cdots$$

Definition: Gauge group $G := (\mathfrak{g}_0, \mathrm{BCH}, 0)$

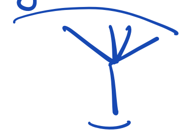
$\circlearrowleft \lambda.\alpha = e^{\mathrm{ad}_\lambda}(\alpha)$

$MC(\mathfrak{g})$

III - The operadic case

Motivation: Deformation theory of $\mathcal{P}_\infty$-algebras $\quad$ $\mathcal{P}$: operad

operations:



↳ **Examples:** associative algebras, commutative algebras, Lie algebras, Poisson algebras, Gerstenhaber algebras, pre-Lie algebras, post-Lie algebras, Batalin-Vilkovisky algebras .....

general construction: "operadic convolution algebra"

$$\mathfrak{g}_{\mathcal{P},A} := \left(\mathrm{Hom}(\mathcal{P}^{¡}; \mathrm{End}A), d, \star\right)$$

Koszul dual cooperad

pre Lie $\Rightarrow$ Lie bracket

ξ

Pre-Lie exponential map: $\mathfrak{g} \underset{\text{“}\ln\text{”}}{\overset{\exp}{\rightleftarrows}} 1\!\!1+\mathfrak{g}$ $\quad\cong$

$$\lambda \longmapsto \exp(\lambda) := 1\!\!1+\lambda+\frac{1}{2}\lambda\star\lambda+\frac{1}{6}(\lambda\star\lambda)\star\lambda$$

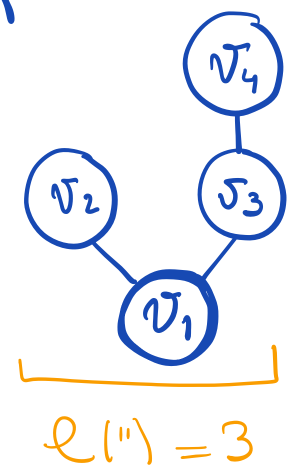
$$+\frac{1}{24}((\lambda\star\lambda)\star\lambda)\star\lambda+\cdots$$

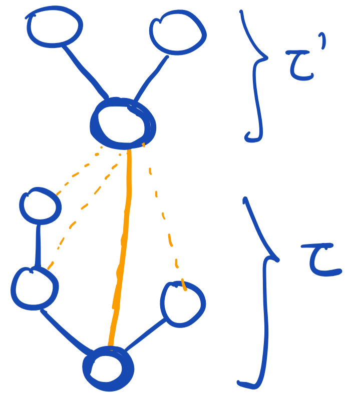
isomorphism where

the "preLie logarithm" = Magnus expansion map.

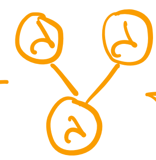
Theorem [Chapoton–Livernet, 2001]

Free pre-Lie algebra on V : rooted trees (RT)



$\tau \star \tau' := \sum_{\text{graftings}}$ 

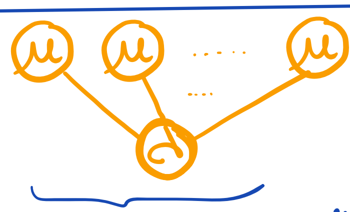
Proposition [Dotsenko–Shadrin–V. 2015].

$$\exp(\lambda) = \mathbb{1} + \sum_{\tau \in RT} \frac{\ell_\tau}{|\tau|!}\,\tau(\lambda) = \mathbb{1} + \lambda + \frac{1}{2}$$  $$+ \frac{1}{6}$$  $$+ \cdots$$

number of levelisations

number of vertices.

Definition:

$$(\mathbb{1}+\lambda) \circledcirc (\mathbb{1}+\mu) := \mathbb{1} + \sum_{n \geq 1} \frac{1}{n!}$$ 

Theorem [Dotsenko–Shadrin–V. 2015].

$(\mathfrak{g}, \star)$ (nilpotent, complete) pre-Lie algebra

$$(\mathfrak{g}_0, BCH, 0) \underset{\exp}{\cong} (\mathbb{1} + \mathfrak{g}_0, \circledcirc, \mathbb{1})$$

$$\lambda . \alpha = (\exp(\lambda) \star \alpha) \circledcirc \exp(-\lambda)$$

Corollary [Dotsenko–Shadrin–V. 2015]

$$\boxed{\mathrm{Deligne}(\mathfrak{g}_{P,A}) \cong (\underbrace{\infty\text{-isotopies}}_{\text{gauges}}, \circ, \mathrm{id})}$$

objects = P_∞-alg

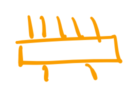
Applications [Campos–Petersen–Robert-Nicoud–Wierstra 2019].

- The universal enveloping algebra detects isomorphisms of (homotopy complete) Lie algebras.
- commutative formality $\Longleftrightarrow$ associative formality
- $C^\bullet_{\mathrm{sing}}(X, \mathbb{Q})$ detects the rational homotopy type of X

IV _ The properadic case

Motivation: Deformation theory of $\mathcal{P}_\infty$-bialgebras $\mathcal{P}$: properad

many inputs

operations: 

many outputs

→ **Examples**: "associative/commutative bialgebras", Frobenius bialgebras, (involutive) Lie bialgebras, infinitesimal bialgebras, double Poisson gebras, pre-Calabi-Yau gebras, etc....

general construction: "properadic convolution algebra"

$$\mathfrak{g}_{P,A} := \big(\mathrm{Hom}(\mathcal{P}^{¡}; \mathrm{End}A), \partial, \star\big)$$

Koszul dual coproperad

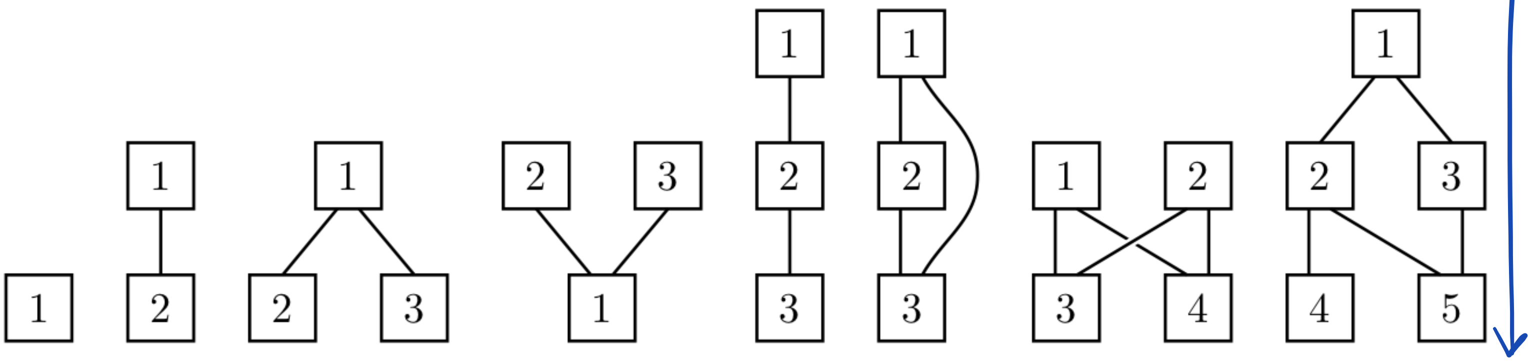
Lie admissible $\Rightarrow$ Lie bracket

definition

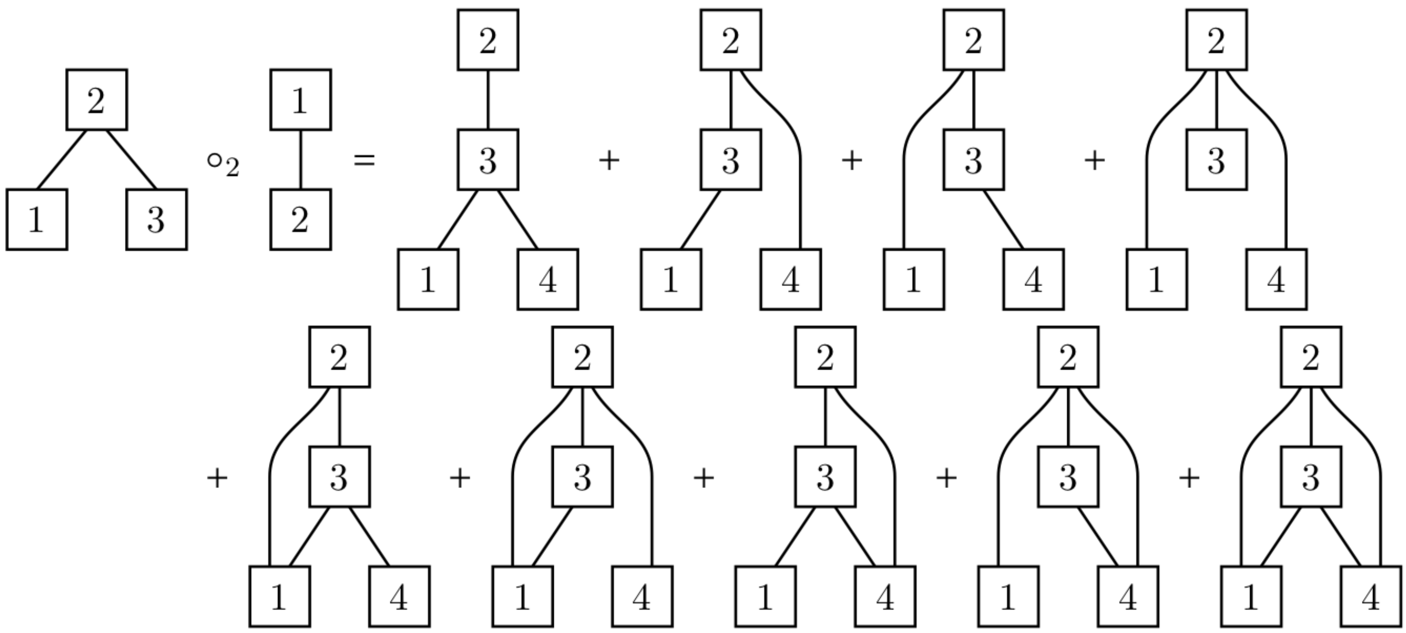
(?) Lie-admissible exponential $\Leftarrow$ iterations of $\star$ (?) **NO!**

<u>Definition</u> [Campos–V.]

The <u>Lie-graph operad</u> : directed simple graphs



Operadic composition : insertion at a vertex + all possible graftings



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morphisms of operads: $\mathrm{Lie} \rightarrowtail \text{Lie-adm} \to \text{Lie-gra} \twoheadrightarrow \text{pre-Lie} \twoheadrightarrow \mathrm{Ass}\,,$

$\Downarrow$

assoc alg $\subset$ pre-Lie alg $\subset$ $\boxed{\text{Lie-graph alg}} \xrightarrow{\substack{\boxed{2}\\ \boxed{1}}}$ Lie-admissible alg $\longrightarrow$ Lie alg

Remark: Lie-adm $\not\subset$ Lie-gra

NOT finitely generated!

$\Updownarrow$

No definition of Lie-graph algebras with a finite number of structure operations

$\hookrightarrow$ operadic calculus mandatory

Proposition [Campos-V.]

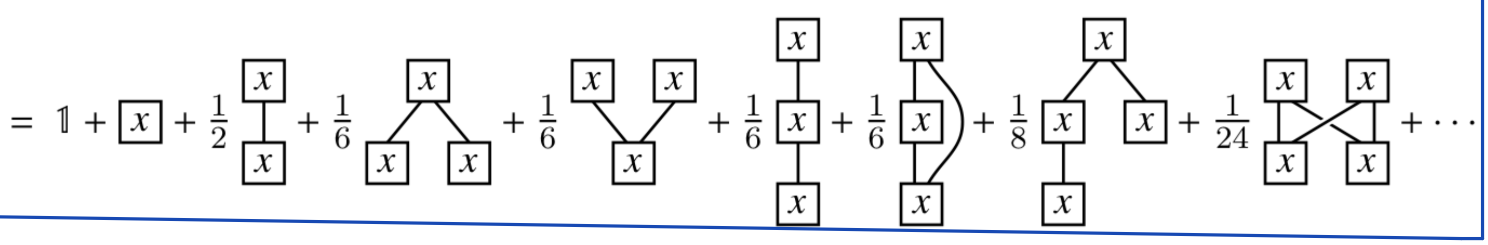
$\mathfrak{g}_{P,A}$ extends to a Lie-graph algebra

Definition Lie-graph exponential map

$$\exp(x) := \mathbb{1} + \sum_{g \in \mathrm{dsGra}} \frac{\ell_g}{|g|!}\, g(x)$$

number of levelisations

number of vertices.

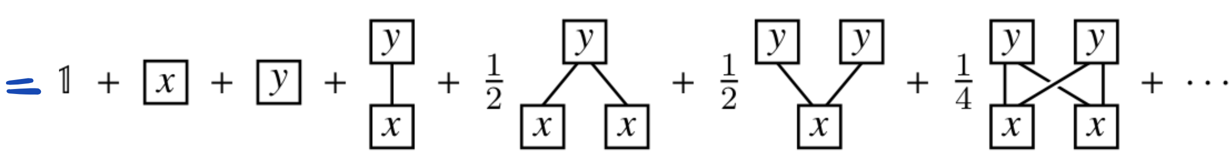


Remark: $\exists$ Lie-graph logarithm inverse

Definition:

$$(\mathbb{1} + x) \circledcirc (\mathbb{1} + y) := \mathbb{1} + \sum_{g \in 2\text{-dsGra}_{\mathbb{S}}} \frac{1}{|\mathrm{Aut}(g)|}\, g(x, y),$$

2 levels



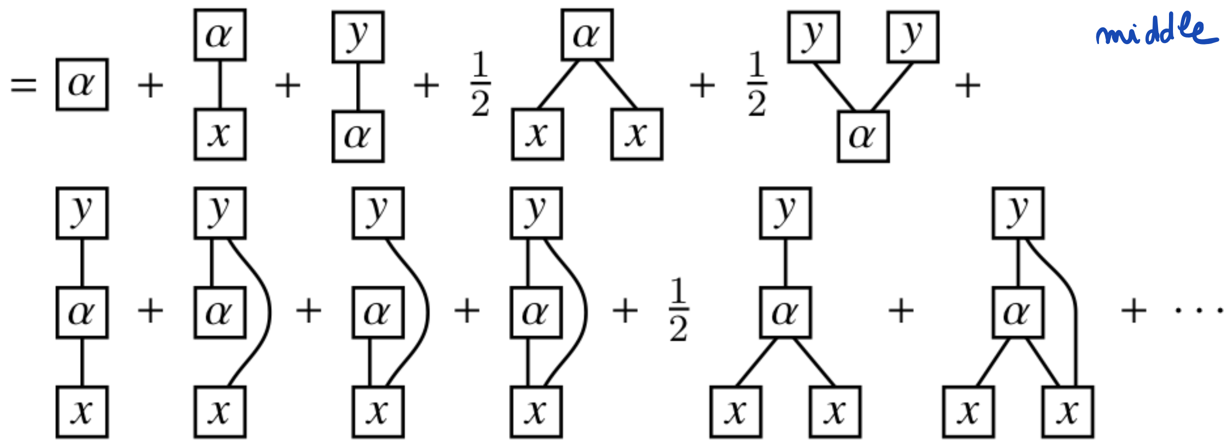
Theorem [Campos-V.]

$(\mathfrak{g}, \star)$ (nilpotent, complete) Lie-graph algebra

$$(\mathfrak{g}_0, BCH, 0) \underset{\exp}{\cong} (\mathbb{1}+\mathfrak{g}_0, \circledcirc, \mathbb{1})$$

$$\lambda . \alpha = (\mathbb{1}+x) \overset{\alpha}{\bowtie} (\mathbb{1}+y) := \sum_{g \in \bowtie\text{-dsGra}_S} \frac{1}{|\mathrm{Aut}(g)|}\, g(x, \alpha, y)$$

3 levels
1 vertex in the middle



Corollary [Campos-V.]

objects = P_∞-bialg.

$$\mathrm{Deligne}(\mathfrak{g}_{P,A}) \cong (\infty\text{-isotopies}, \circ, \mathrm{id})$$

V _ The higher case

assoc alg ⊂ pre-Lie alg ⊂ Lie-graph → Lie alg ⊂ L_∞-alg

classical exponential — tree exponential — graph exponential — BCH — higher BCH

Remark: L_∞-alg simpler than Lie alg

$[\,,\,] \longrightarrow \ell_2, \ell_3, \ell_4, \ldots \ell_n, \ldots$

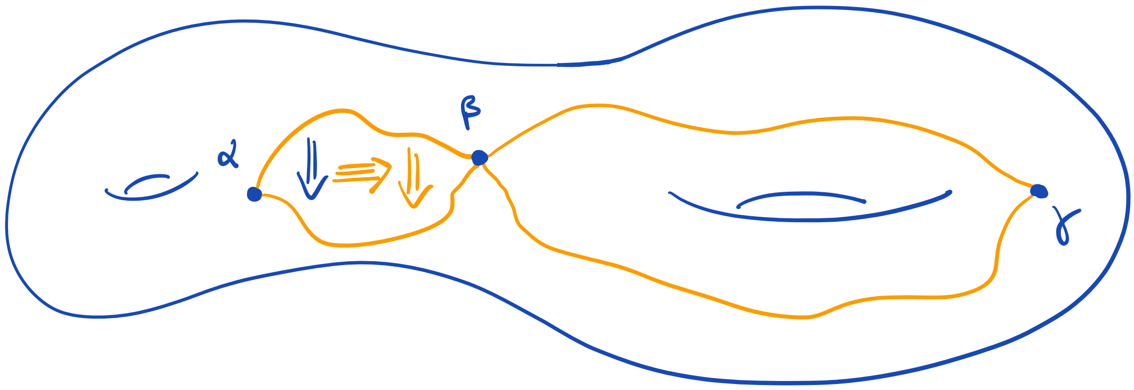
$\mathfrak{g}^{\otimes 2} \to \mathfrak{g}$ $\qquad$ $\mathfrak{g}^{\otimes n} \to \mathfrak{g}$

↑ homotopy for the Jacobi identity

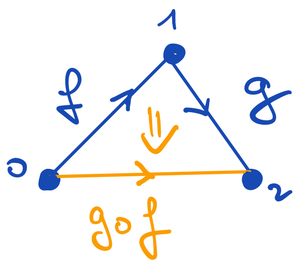
ℓ_2 does not satisfy the Jacobi relation $\Longrightarrow$ ~~group~~

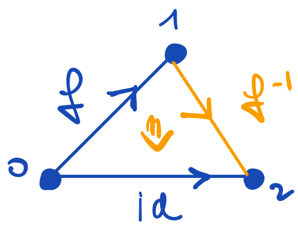
but up to homotopy $\Longrightarrow$ $\boxed{\infty\text{-groupoid}}$

<u>Heuristically</u>: ∞-groupoid $=$ Topological space



<u>model</u>: Kan complex := simplicial set $\exists$ horn fillers



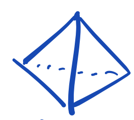


$\mathfrak{g}$ (complete) $\mathcal{L}_\infty$-algebra

<u>Solution 1</u> [Sullivan 1977]

[Hinich 1997]

$$MC_\bullet(\mathfrak{g}) := MC\big(\mathfrak{g} \otimes \Omega_{poly}(\Delta^\bullet)\big) \in \text{Kan complexes} \subset sSet$$

Δ^n: geometrical n-simplex 

polynomial differential forms

simplicial $\left\{ \underbrace{\underbrace{\mathcal{L}_\infty\text{-alg}}_{} \otimes \underbrace{\text{com alg}}_{}}_{\mathcal{L}_\infty\text{-alg}} \right.$

Maurer–Cartan equation: $d\alpha + \frac{1}{2}\ell_2(\alpha,\alpha) + \frac{1}{6}\ell_3(\alpha,\alpha,\alpha) + \cdots$

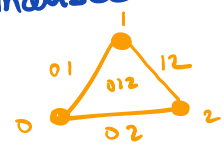
• <u>$n=0$</u>: $MC_0(\mathfrak{g}) = MC(\mathfrak{g} \otimes \mathbb{Q}) = MC(\mathfrak{g})$: OK!

• $\underline{n=1}$: $MC_1(\mathfrak{g}) \not\ni$ (algebraic) gauges

$\Downarrow$ need to refine

Definition: Dupont simplicial contraction

$$h_\bullet \circlearrowright \underbrace{\Omega_{poly}(\Delta^\bullet)}_{\dim = +\infty} \underset{i_\bullet}{\overset{p_\bullet}{\rightleftarrows}} \underbrace{N^*(\Delta^\bullet)}_{\dim < +\infty}$$: normalised cochains



Definition-Proposition [Getzler 2009]

$$\gamma_\bullet(\mathfrak{g}) := MC_\bullet(\mathfrak{g}) \cap \underbrace{\ker h_\bullet}_{\text{gauge condition}}$$ $\in$ Kan complexes

$$\gamma_\bullet(\mathfrak{g}) \overset{\sim}{\hookrightarrow} MC_\bullet(\mathfrak{g})$$

• $\underline{n=0}$: $\gamma_0(\mathfrak{g}) = MC(\mathfrak{g})$

• $\underline{n=1}$: $\gamma_1(\mathfrak{g})$ = algebraic gauges !

Problem: implicit definition . . .

Definition [Robert-Nicoud-V.]

$$mc^\bullet := \widehat{\mathcal{L}_\infty}(\underbrace{N_*(\Delta^\bullet)}_{\text{cosimplicial } \mathcal{C}om_\infty\text{-alg}})$$: universal cosimplicial complete $\mathcal{L}_\infty$-alg : explicit formulas

Theorem [Robert-Nicoud-V.]

$\gamma_\bullet(\mathfrak{g}) \cong \mathrm{Hom}_{\mathcal{L}_\infty\text{-alg}}(\mathrm{mc}^\bullet, \mathfrak{g})$ representation functor

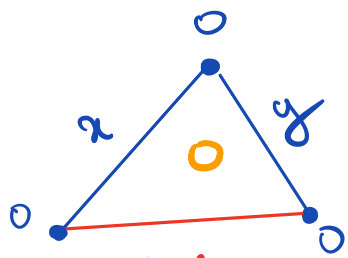
algebraic Kan complex

with canonical & explicit horn fillers (by 0)

$=$ higher BCH formulas

First example:

$\mathfrak{g}$ Lie alg



BCH(x,y) : new form!

General formula

Proposition 5.10. *The higher BCH product is given by*

$$\Gamma^n_k(x;y) = \sum_{\substack{\tau\in \mathrm{PaPRT}\\ \chi\in \mathrm{Lab}^{[n],k}(\tau)}} \prod_{\substack{\beta \text{ block of } \tau\\ \lambda^{\beta(\chi)}_{[n]}\neq 0}} \frac{(-1)^k}{\lambda^{\beta(\chi)}_{[n]}[\beta]!}\, \ell_\tau\left(x_{\chi(1)},\dots,x_{\chi(p)}; (-1)^k \mathrm{d}y - \sum_{l\neq k}(-1)^{k+l}x_{\hat{l}}\right),$$

The story continues:

assoc alg $\subset$ pre-Lie alg $\subset$ Lie-graph $\Rightarrow$ Lie alg $\subset$ $\mathcal{L}_\infty$-alg $\subset$ curved / partition / absolute $\mathcal{L}_\infty$-alg

[Roca Lucio 2023]

$\uparrow$

char $\mathbb{k}>0$, curvature, etc...

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Maurer–Cartan Methods in Deformation Theory

The Twisting Procedure

Vladimir Dotsenko, Sergey Shadrin
and Bruno Vallette

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